Quantum resonances and rectification of driven cold atoms in optical lattices

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Classical Hamiltonian ratchets have been recently successfully realized using cold atoms in driven optical lattices. Here we study the current rectification of the motion of a quantum particle in a periodic potential exposed to an external ac field. The dc current appears due to the desymmetrization of Floquet eigenstates, which become transporting. Quantum dynamics enhances the dependence of the current on the initial phase of the driving field. By changing the laser field parameters which control the degree of space-time asymmetry, Floquet eigenstates are tuned through avoided crossings. These quantum resonances induce resonant changes of the resulting current. The width, strength and position of these quantum resonances are tunable using control parameters of the experimental realization with cold atoms.

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The *ratchet* effect, i.e. the possibility to rectify transport with the help of zero-mean perturbations, has been discussed in order to explain mechanisms of microbiological motility [1], and was applied to other situations as well [2], including quantum systems [3]. The dissipationless limit of Hamiltonian ratchets [4, 5, 6, 7] and the corresponding symmetry predictions [4] have been recently successfully studied with cold Rubidium and Cesium atoms in optical lattices with a two-harmonics driving and a tunable weak dissipation [8]. In these experiments, the mechanism of the Sisyphus cooling [9] has been used in order to furnish initial conditions in form of an *optical lattice*: an ensemble of atoms localized in the wells of a periodic potential. In the momentum space this corresponds to a narrow distribution near the momentum $p = 0$. This is essential for the observation of the rectification effect, since for broad initial distributions the asymptotic current tends to zero.

The dissipationless case can be readily implemented in atom optics by using laser beams which generate far detuned standing waves [10]. Appropriate time-dependent forces can be applied to the atoms by phase modulating the lattice beams [8]. The reachable quantum regime of cold atoms becoming coherent matter waves calls for a study of a quantum Hamiltonian ratchet. Previous studies of quantum ratchets were based on the kicked rotor model [3, 7, 11], which is easily treated numerically, but posesses a broad frequency spectrum of the kick drive. The above mentioned experimental realization of a two-frequency driven classical Hamiltonian ratchet instead suggests to consider the corresponding quantum problem of a particle moving in a spatially periodic potential under the influence of a two frequency ac force of zero mean and to search for quantum peculiarities of the current rectification.

The Hamiltonian for a particle with position x and momentum p is [8]

$$H(x,p,t) = \frac{p^2}{2} + (1+\cos(x)) - xE(t), \qquad (1)$$

where $E(t)$ is an external periodic field of zero mean, $E(t+T) = E(t)$, $\int_0^T E(t)dt = 0$. For the classical case there are two symmetries which need to be broken to fulfill the necessary conditions for a nonzero dc current [4]. If $E(t) = -E(t+T/2)$ is shift symmetric, then (1) is invariant under symmetry

$$S_a : (x,p,t) \to (-x,-p,t+T/2). \qquad (2)$$

If $E(t) = E(-t)$ is symmetric, then (1) is invariant under

$$S_b : (x,p,t) \to (x,-p,-t). \qquad (3)$$

The phase space of system (1) is mixed, containing both regular regions and a chaotic layer around $p = 0$. Whenever the system (1) possesses any of the two symmetries S_a and/or S_b, directed transport is forbidden inside the chaotic layer [4].

The two frequency driving

$$E(t,t_0) = E_1\cos(\omega(t-t_0)) + E_2\cos(2\omega(t-t_0)+\theta) \quad (4)$$

ensures that for $E_1, E_2 \neq 0$ S_a is always violated. In addition S_b is violated for $\theta \neq 0, \pm\pi$. The appearance of a nonzero dc current $J_{ch} = \lim_{t\to\infty} 1/t \int_{t_0}^t p(t')dt'$ in this case is due to a desymmetrization of the chaotic layer structure (Fig.1a). It induces a desymmetrization of the events of directed motion to the right and left [12]. Due to ergodicity inside the layer, the asymptotic current is independent of the initial phase t_0, for initial conditions located inside the chaotic layer. With the specific choice of the driving (4) it follows $J_{ch}(\theta) = -J_{ch}(-\theta)$ and $J_{ch}(\theta) = -J_{ch}(\theta+\pi)$ [4]. From perturbation theory it follows $J_{ch} \sim E_1^2 E_2 \sin\theta$ [4, 5]. An efficient sum rule allows to compute the average current J_{ch} by proper integration over the chaotic layer [7].

The Hamiltonian (1) is periodic in time with period T. The solutions $|\psi_\alpha(t)\rangle = U(t,t_0)|\psi_\alpha(0)\rangle$ of the

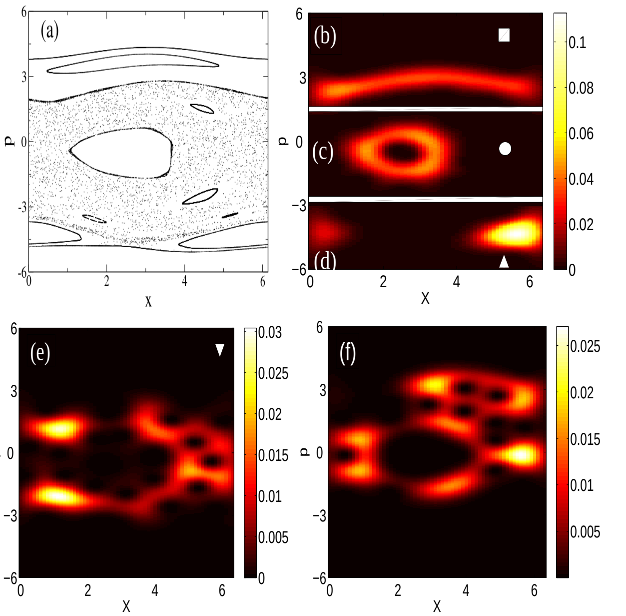

FIG. 1: (a) Poincaré section for the classical limit; (b-f) Husimi representations for different Floquet eigenstates for the Hamiltonian (1) with $\hbar = 0.2$ (momentum is in units of the recoil momentum, $p_r = \hbar k_L$, with $k_L = 1$). The parameters are $E_1 = E_2 = 2$, $\omega = 2$, $\theta = -\pi/2$ and $t_0 = 0$ for (b-e), and $E_1 = 3.26$, $E_2 = 1$, $\omega = 3$, $\theta = -\pi/2$ and $t_0 = 0$ for (f).

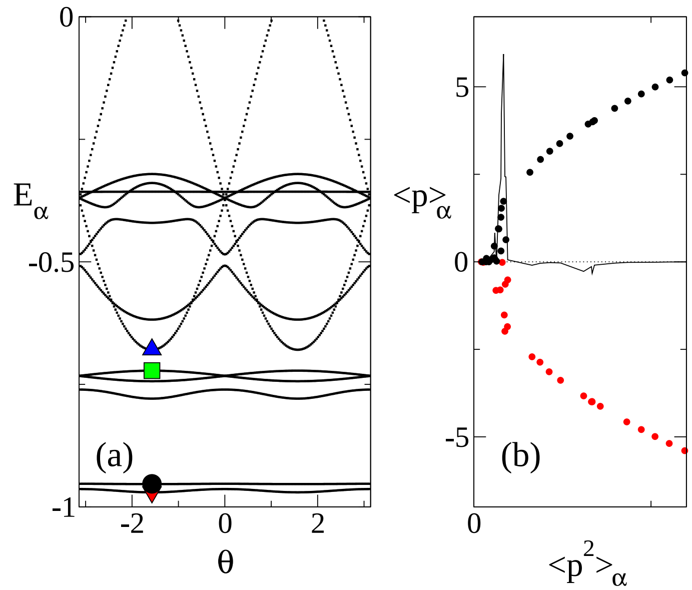

FIG. 2: (a) A part of the quasienergy spectrum as a function of the parameter θ. The symbols indicate the corresponding Floquet states shown in Fig.1(b-e). (b) The mean momentum $\langle p\rangle_\alpha$ vs mean kinetic energy $\langle p^2\rangle_\alpha$ for the Floquet states (filled circles). The line corresponds to the cumulative momentum, P_α (see text). The parameters are the same as in Fig.1(b-e). All the momenta are scaled in units of the recoil momentum p_r.

Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H(t,t_0)|\psi(t)\rangle, \tag{5}$$

can be characterized by the eigenfunctions of $U(t_0 + T, t_0)$ which satisfy the Floquet theorem: $|\psi_\alpha(t)\rangle = e^{-i\frac{E_\alpha}{T}t}|\phi_\alpha(t)\rangle$, $|\phi_\alpha(t+T)\rangle = |\phi_\alpha(t)\rangle$. The quasienergies E_α $(-\pi < E_\alpha < \pi)$ and the Floquet eigenstates can be obtained as solutions of the eigenvalue problem of the Floquet operator

$$U(T,t_0)|\phi_\alpha(t_0)\rangle = e^{-iE_\alpha}|\phi_\alpha(t_0)\rangle. \tag{6}$$

The Floquet eigenstates provide a complete orthonormal basis and the stroboscopic quantum state can be expressed as

$$|\psi(mT,t_0)\rangle = \sum_\alpha C_\alpha(t_0)e^{-imE_\alpha}|\phi_\alpha(t_0)\rangle, \tag{7}$$

where the coefficients $\{C_\alpha\}$ depend on t_0. For later convenience the integer α, which sorts the states $|\phi_\alpha\rangle$ such that the mean kinetic energy $\langle p^2\rangle_\alpha \equiv 1/T\int_0^T\langle\phi_\alpha|\hat{p}^2|\phi_\alpha\rangle dt_0$ monotonically increases.

With the help of a gauge transformation, $|\psi\rangle \to \exp(-\frac{i}{\hbar}x\int_0^t E(t')dt')|\psi\rangle$ [13], the solution of the time-dependent Schrödinger equation for the Hamiltonian (1), may be written as

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\int_0^t\{\frac{1}{2}[\hat{p}-A(t',t_0)]^2+(1+\cos x)\}dt'}|\psi(0)\rangle, \tag{8}$$

with the vector potential $A(t,t_0) = -\frac{E_1}{\omega}\sin(\omega(t-t_0)) - \frac{E_2}{2\omega}\sin(2\omega(t-t_0)+\theta)$. Due to discrete translational invariance and Bloch's theorem all Floquet states are characterized by a quasimomentum κ with $|\phi_\alpha(x+2\pi)\rangle = e^{i\hbar\kappa}|\phi_\alpha(x)\rangle$. Here we choose $\kappa = 0$ which corresponds to initial states where atoms equally populate all (or many) wells of the spatial potential. The wave function is expanded in the plane wave eigenbasis of the momentum operator $\hat{p}$, $|n\rangle = \frac{1}{\sqrt{2\pi}}e^{inx}$. The Floquet propagator $U(T,t_0)$ is obtained by solving the Schrödinger equation over a single period T for a sufficiently large set of plane waves $|n\rangle$ with $n = 0, \pm1, \pm2, ..., \pm N$, where $2N+1$ is the total number of basis states taken into account. The numerical calculations follow the integration method described in Ref.[14], results are shown for $N = 60$, which do not depend upon further increase of N.

If the Hamiltonian is invariant under the shift symmetry S_a (2), then the Floquet matrix has the property $U(T,t_0) = \left[U^{\maltese}(T/4,t_0)\,U(T/4,t_0)\right]^T U^{\maltese}(T/4,t_0)\,U(T/4,t_0)$ [15]. Here $U^{\maltese}$ performs a transposition along the codiagonal of U. In our case S_a is always violated.

If the Hamiltonian is invariant under the time reversal symmetry S_b (3), then the Floquet matrix has the property $U(T,t_0) = U(T,t_0)^{\maltese}$ [15]. That symmetry will be recovered for $\theta = 0, \pi$. Then the Floquet matrix has an irreducible representation using even and odd basis

states $|n\rangle_{s,a} = (|n\rangle \pm |-n\rangle)/\sqrt{2-\delta_{n,0}}$.

The mean momentum expectation value $J(t_0) = \lim_{t\to\infty} 1/t \int_{t_0}^{t} \langle\psi(t,t_0)|\hat{p}|\psi(t,t_0)\rangle$ measures the asymptotic current. Expanding the wave function over the Floquet states the current becomes

$$J(t_0) = \sum_{\alpha} \langle p\rangle_\alpha |C_\alpha(t_0)|^2, \qquad (9)$$

where $\langle p\rangle_\alpha$ is the mean momentum of the Floquet state $|\phi_\alpha\rangle$.

For the symmetric case $\theta = 0, \pm\pi$ it follows that $\langle p\rangle_\alpha = 0$ for all α. Consequently $J = 0$ in this case. We especially note that this is true for states with arbitrarily large kinetic energy, for which the corresponding quasienergies become almost pairwise degenerated. For $\theta \neq 0, \pm\pi$ the Floquet states become asymmetric. The quasidegeneracies are removed (Fig.2a). Especially Floquet states with large kinetic energies acquire large mean momenta (Fig.2b), thus becoming transporting. This is a consequence of the fact that the true perturbation parameter regulating the desymmetrization around the symmetric quasidegeneracy points for small θ is proportional to $nE_1^2E_2\theta$. Fig.2b also shows the cumulative average momentum, $P_{\alpha+2} = P_\alpha + \langle p\rangle_{\alpha+1} + \langle p\rangle_{\alpha+2}$, $P_0 = \langle p\rangle_0$. The asymmetry stems mainly from Floquet states located in the chaotic layer region of the classical phase space (see results below). With increasing $\langle p^2\rangle_\alpha$, P_α goes to zero in full accordance with the fact that total current over the whole momentum space should be zero [7].

In Fig.1(b-e) we present Husimi distributions [16] for several Floquet states. The dimensionless Planck constant $\hbar = 0.2$ is in a range, when it is possible to establish a correspondence between different Floquet states and the invariant manifolds of the mixed phase space for the classical limit. Each plot carries a symbol, which shows the corresponding location of the quasienergy of that state in Fig.2a. The states (b-d) are located in various regular phase space regions, while state (e) is located inside the chaotic layer.

The current for an initial condition $|\psi(t_0)\rangle$ depends in general on the initial phase t_0. Note that in the classical case such an initial condition may also lead to some dependence of the classical current on t_0, since the initial distribution may overlap with different regular transporting manifolds. However, if we start with a cloud of particles exactly located inside the chaotic layer, the asymptotic current will be independent of t_0 for any choice of the distribution function over the chaotic manifold. This is not true for the quantum case where the current may even change its sign with the variation of t_0. It is a consequence of the linear character of the Schrödinger equation [17]. We will first discuss the results obtained after averaging over the initial phase t_0. Then we can assign a unique current value, $J = 1/T \int_0^T J(t_0)dt_0$, for fixed parameters of the ac-field, E_1, E_2, and θ.

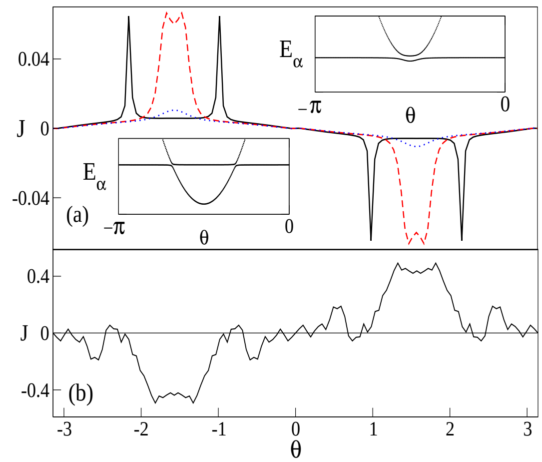

FIG. 3: (a) The average current J (in units of the recoil momentum) vs θ for different amplitude values of the second harmonic, E_2: 0.95 (pointed line), 1 (dashed line) and 1.2 (solid line). Insets: relevant details of the quasienergy spectrum versus θ in the resonance region for $E_2 = 1$ (top right) and $E_2 = 1.2$ (bottom left). The parameters are $E_1 = 3.26$ and $\omega = 3$. (b) The average current J (in units of the recoil momentum) vs θ for $E_1 = 3$, $E_2 = 1.5$ and $\omega = 1$.

Figure 3(a) shows the dependence of the average current on the asymmetry parameter θ for the initial condition $|\psi\rangle = |0\rangle$. The average current J shows the expected symmetry properties $J(\theta) = -J(\theta+\pi) = -J(-\theta)$. On top of the smooth curves we find several resonant peaks for $E_2 = 0.95$ where the current value changes drastically. Comparing with the quasienergy spectrum, these resonances can be unambiguously associated with avoided crossings between two Floquet eigenstates. The Husimi distributions show that one state locates in the chaotic layer, and another one in a transporting island. Off resonance the initial state mainly overlaps with the chaotic state, which yields some nonzero current. In resonance Floquet states mix, and thus the new eigenstates contain contributions both from the original chaotic state as well as from the regular transporting island state. The Husimi distribution of the mixed state is shown in Fig.1f, the strong asymmetry is clearly observed. The regular island state has a much larger current contribution, resulting in a strong enhancement of the current.

¿From an experimental point of view a too narrow resonance may become undetectable due to resolution limitations. We thus studied how to vary the width of the resonance without much affecting its amplitude. It turns out to be possible by tuning another control parameter, e.g. the amplitude E_2. We increase this field amplitude in order to disentangle the two Floquet states and remove the avoided crossing. That will happen for some value of E_2 at $\theta = \pm\pi/2$. The details of the quasienergy spectrum around that critical point are shown in the insets in Fig.3a. The two quasienergy spectra disentangle for $E_2 = 1$ but stay close over a sufficient broader range of θ

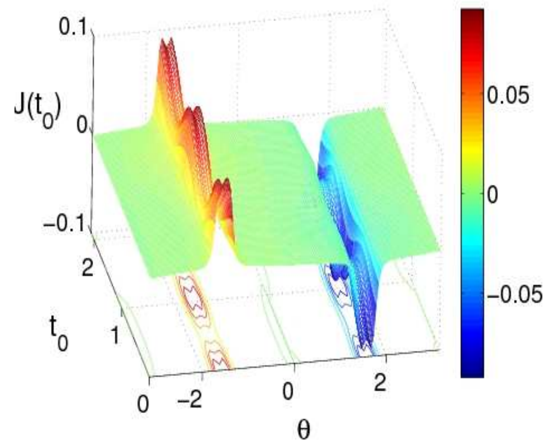

FIG. 4: Current dependence on the intial phase t_0 and θ. The parameters are the same as in Fig.1f.

values. Thus the resonances become broader, as seen in Fig.3a. Further increase of E_2 to a value of 1.2 leads to a strong separation of the two spectra, and consequently to a fast decay of the amplitude of the resonance.

The above numerical results for $\omega = 2$ show a maximum current value in the resonance region of the order of 0.06 in units of recoil momenta. In order to increase that output, we drive the system into strongest resonance by choosing $\omega = 1$, since that driver frequency matches the oscillation frequencies of particles at the bottom of the spatial periodic potential. The result is shown in Fig.3b. We again observe a clear broad resonance, but the maximum current value increases by an order of magnitude up to 0.5 in units of recoil momenta.

As already mentioned, without averaging over t_0, the current depends on the initial phase. However the observed resonance structures are due to resonant interaction between Floquet states, or avoided crossings of quasienergies. These resonances are independent of the initial phase t_0. Indeed, in Fig.4 we plot the nonaveraged current as a function of both θ and t_0. While the smooth background is barely resolvable with the naked eye, the resonances are clearly seen, and their position is not depending on t_0, while their amplitude does. That implies that one can further maximize the resonant current by choosing proper initial phases t_0, reaching values above the recoil momentum.

Note that our approach is very different from a recently proposed modified kicked rotor model [18], where a biased *acceleration* appears for an initial condition with preassigned nonzero velocity. Here we study the regime of *stationary* current for the general case of an initial state of zero momentum for a model which has a finite current also in its classical counterpart [6, 12].

In summary, we have studied the mechanisms of average current appearance in driven quantum systems with broken symmetries. The key source of such directed transport is the desymmetrization of Floquet states. A peculiarity of the quantum ratchet is the strong dependence of the current on the initial phase of the applied field. Moreover, we found quantum resonances induced by avoided crossings between Floquet states which enhance the current drastically. Optimizing the drive frequency, amplitude and initial phase, resonant currents easily reach the recoil momentum value and should be experimentally observable using driven cold atoms in optical lattices.

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